A sign-problem-free effective model of triangular lattice antiferromagnetism

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Outline of talk

- Geeral discussion of quantum antiferromagnets and sign problems
- Derivation of effective model - bosonic spinons coupled to *odd* $\mathbb{Z}_2$ gauge field
- Sign-problem-free mapping - generalized particle/vortex dualities

Frustrated antiferromagnetism leads to exotic quantum phases

$H = \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$, trivial phase for half-integer spin precluded by LSM theorem

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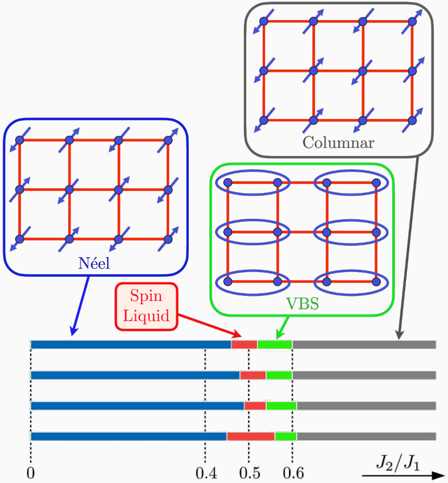

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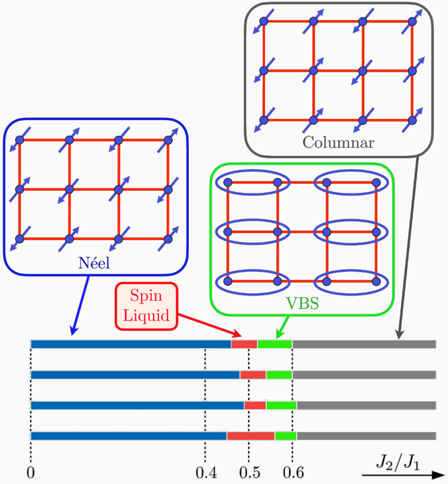

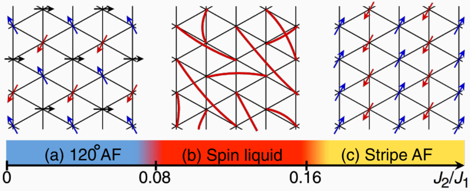

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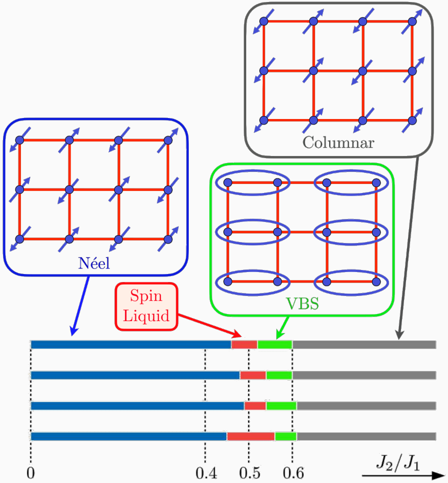

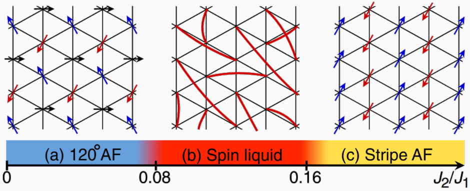

Numerics generally restricted to small system sizes, resolution of ordered phases difficult

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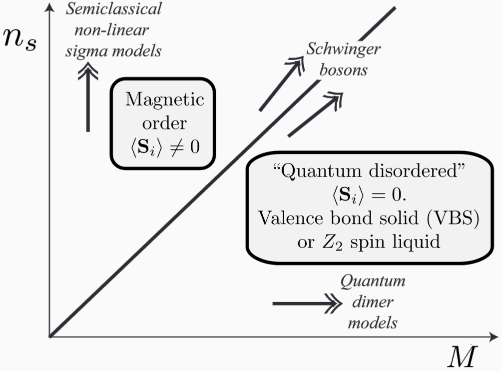
"Partition function-based" numerics (PIMC, DQMC, SSE...)

- Direct evaluation of $\mathrm{tr}\left[e^{-\beta H}\right]$
- Stochastic methods allow for much larger system sizes
- "Sign problem" prevents generic applicability
- Can cure sign problem through clever mappings, designer Hamiltonians, etc

Bosonic spinon description captures orderd phases

Rewrite spin model using $\mathbf{S} = \frac{1}{2} s^\dagger_{i\alpha} \sigma^\alpha_\beta s^\beta_i$, where $\alpha = 1 \dots M$ and $s^\dagger_{i\alpha} s^\alpha_i = n_s$ allows for a large n_s, M study.



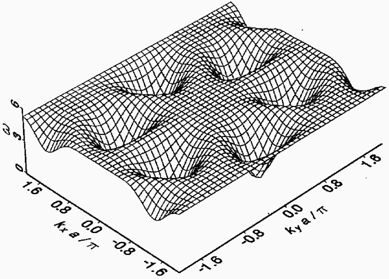
$$\mathcal{L} = \sum_i \left[s^\dagger_{i\alpha} \left(\frac{\partial}{\partial \tau} + i\lambda_i \right) s^\alpha_i - i\lambda_i n_s \right]$$

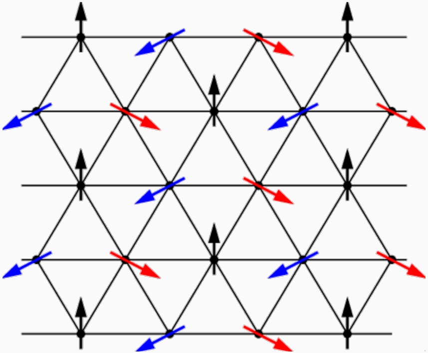
$$- \sum_{ij} \frac{J_{ij}}{2M} \left(\epsilon^{\alpha\beta} s^\dagger_{i\alpha} s^\dagger_{j\beta} \right) \left(\epsilon_{\gamma\delta} s^\gamma_i s^\delta_j \right)$$

Find saddle-point solutions for $\mathcal{Q}_{ij} \equiv \langle \epsilon_{\alpha\beta} s^\alpha_i s^\beta_j \rangle$, λ_i

Sachdev 1992.

Mean-field spinon solution supports gapped vison excitations





Spinon condensation → magnetic order

Sachdev 1992; Huh, Punk, and Sachdev 2011.

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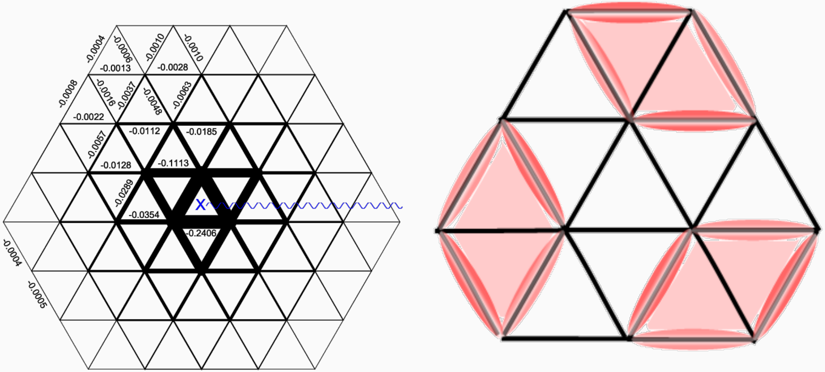

Vison condensation $\rightarrow$ VBS order

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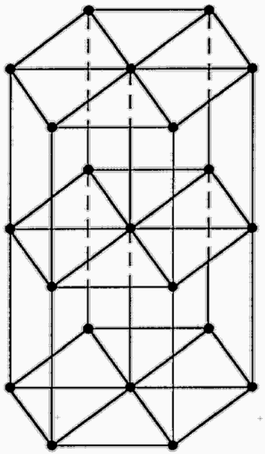
Effective model of triangular lattice antiferromagnetism

$$\mathcal{Z} = \sum_{s_{j,j+\hat{\mu}}=\pm 1} \prod_{j,\alpha} \int \mathrm{d}z_{j\alpha}\, \delta\left(\sum_{\alpha=1,2} |z_{j\alpha}^2| - 1\right)$$
$$\times \left[\prod_j s_{j,j+\tau}\right] \exp\left(-H[z_\alpha, s]\right)$$
$$H[z_\alpha, s] = -\frac{J}{2} \sum_{\langle j,\mu \rangle} s_{j,j+\hat{\mu}} \left(z_{j,\alpha}^* z_{j+\hat{\mu},\alpha} + \text{ c.c}\right)$$
$$- K \sum_{\triangle\square} \prod_{\triangle\square} s_{j,j+\hat{\mu}}\,.$$



Jian et al. 2018.

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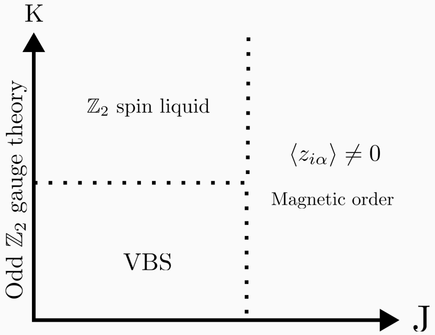

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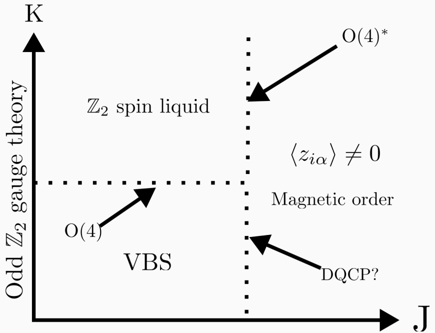

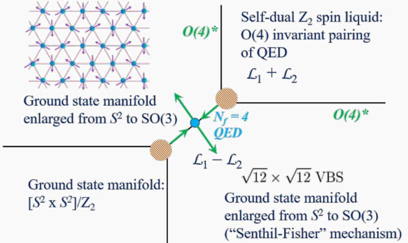

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Mapping of a classical $\mathrm{O}(2)$ model, particle-vortex duality

$$\mathcal{Z} = \prod_i \int_0^{2\pi} \frac{\mathrm{d}\theta_i}{2\pi} \exp\left[-H\right] \quad , \quad H = -\frac{J}{\pi} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) = -\frac{J}{\pi} \sum_{i\mu} \cos\left(\Delta_\mu \theta_i\right)$$

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$$e^{-J(1-\cos(\theta))/\pi} \approx \sum_{n=-\infty}^{\infty} e^{-J(\theta - 2\pi n)^2/(2\pi)} \, ,$$

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$$\sum_{n=-\infty}^{\infty} e^{-J(\theta - 2\pi n)^2/(2\pi)} = \frac{1}{\sqrt{2J}} \sum_{p=-\infty}^{\infty} e^{i\pi p^2/(2J) - ip\theta} ,$$

we get

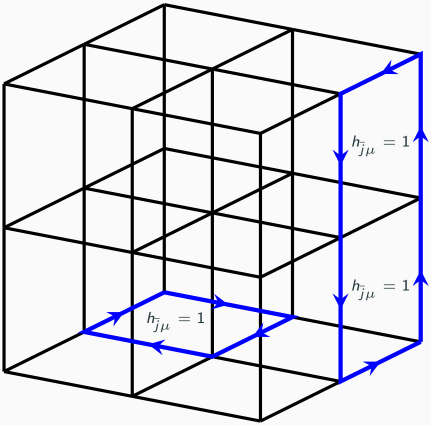
$$\mathcal{Z} = \sum_{p_{i\mu}} \prod_i \int_0^{2\pi} \frac{\mathrm{d}\theta_i}{2\pi} e^{-H} \quad , \quad H = \frac{\pi}{2J} \sum_{i,\mu} p_{i\mu}^2 + i p_{i\mu} \Delta_\mu \theta_i$$

Mapping of a classical O(2) model, particle-vortex duality

Integrating out θ_i enforces $\Delta_\mu p_{i\mu} = 0$.

$p_{i\mu}$ must form *closed current loops*.

Can be enforced by taking $p_{i\mu} = \epsilon_{\mu\nu\lambda}\Delta_\nu h_{\bar{i}\lambda}$



Classical O(2) model with coupling to $\mathbb{Z}_2$ gauge field

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Carefully following s_{ij} through Villain representation gives mutual Chern-Simons coupling between $h_{\bar{i},\mu}$ and s_{ij},

$$i\pi \epsilon_{\mu\nu\lambda} \Delta_\nu h_{\bar{i},\lambda} \frac{1 - s_{i,i+\mu}}{2} = i\pi h_{\bar{i},\mu} \frac{1 - \prod_\square s_{i,i+\mu}}{2}$$

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Claim: integration over s_{ij} can be accomplished by independently summing over possible plaquette values $\prod_{\square} s_{ij} \equiv \Phi_{\bar{i}\mu} = \pm 1$.

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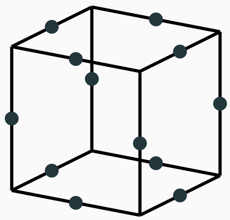

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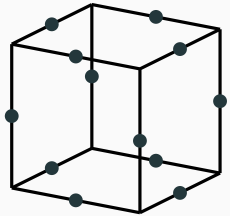
This constraint
is enforced *dynamically* by the redundant degrees of freedom present in the height field representation

$$h_{\bar{i}\mu} \to h_{\bar{i}\mu} + \Delta_\mu f_{\bar{i}}$$

$$\exp\left[i\pi \Delta_\mu f_{\bar{i}} \frac{1 - \Phi_{\bar{i}\mu}}{2}\right] = \exp\left[-i\frac{\pi}{2} f_{\bar{i}} \Delta_\mu \Phi_{\bar{i}\mu}\right]$$



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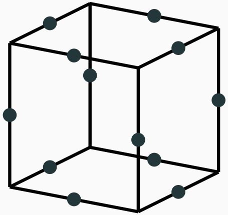

$$h_{\bar{i}\mu} \to h_{\bar{i}\mu} + \Delta_\mu f_{\bar{i}}$$

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Because of this, we are allowed to independently integrate over $\Phi_{\bar{i}\mu} = \pm 1$.

Classical O(2) model with coupling to $\mathbb{Z}_2$ gauge field

$$\mathcal{Z} = \sum_{h=-\infty}^{\infty} \exp\left[-H\right]$$

$$H = \frac{\pi}{2J} \sum_{i,\mu} \left(\epsilon_{\mu\nu\lambda} \Delta_\nu h_{\bar{i},\lambda}\right)^2 + K_d \sum_{i,\mu} \sigma_{i,i+\mu}$$

$$\sigma_{\bar{i},\bar{i}+\mu} \equiv \left(2h_{\bar{i},\mu} - 1\right) \mod 2$$

$$\tanh K_d \equiv e^{-2K}$$

$J \to 0$ limit recovers 3D Ising model - dual to even $\mathbb{Z}_2$ gauge theory.

Accomodation of Berry phase

Berry phase introduces an extra phase factor, $H \to H - i\pi \sum_j \frac{1-s_{j,j+\tau}}{2}$.

Park and Sachdev 2002.

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$$\widetilde{h} \equiv h + h^0 \quad , \quad \epsilon_{\mu\nu\lambda} \Delta_\nu h^0_{\bar{i}\lambda} = \delta_{\mu\tau}$$

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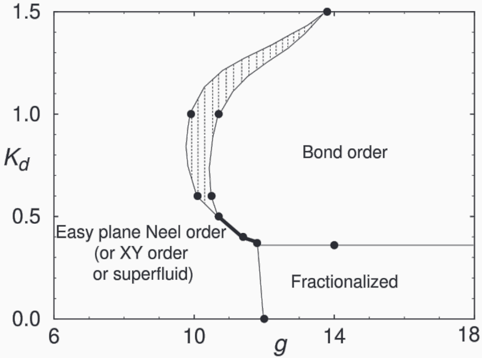
Ising term is now $K_d \sum_{i,\mu} \varepsilon_{i,i+\mu} \sigma_{i,i+\mu}$,
where $\prod_\square \varepsilon_{i,i+\mu} = -1$ for spatial plaquettes



Park and Sachdev 2002.

Elevation to O(4) model

Two complex DOFs, $z_{i\alpha} \equiv r_{i\alpha} e^{i\theta_\alpha}$, constraint $\sum_\alpha r_{i\alpha}^2 = 1$

Hopping $z_{i,\alpha}^* z_{i+\mu,\alpha} + \text{c.c} \to r_{i,\alpha} r_{j,\alpha} \cos(\Delta_\mu \theta_{i,\alpha})$

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Performing Villain approximation on each θ_α breaks O(4) symmetry to O(2) $\otimes$ O(2), must use exact identity

$$e^{J\cos\theta} \propto \sum_{p=-\infty}^{\infty} e^{ip\theta} I_p(J) \quad \ln I_p(J \gg 1) \approx \frac{2}{J} p^2$$

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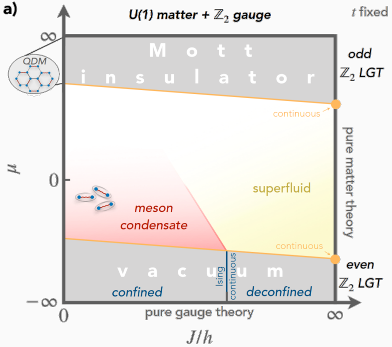
$$H[r_\alpha, h_\alpha] = \sum_{\langle j,\mu\rangle} \left[-\ln I_{p_{j,\alpha,\mu}}(J r_{j,\alpha} r_{j+\widehat{\mu},\alpha}) + K_d \varepsilon_{\bar{j},\mu} \sigma_{\bar{j},\mu}\right]$$

Bose-Hubbard model in $2+1$D coupled to $\mathbb{Z}_2$ gauge field - what happens at non-integer filling?

$$H = J\sum_{i\lambda} s_{i,i+\lambda}\cos\left(\Delta_\lambda\theta_i + i\mu\delta_{\lambda\tau}\right) + K\sum_{\square}\prod_{\square} s_{ij}$$

$$\Rightarrow \frac{1}{2J}\sum_{i\lambda}\left(p_{i\lambda} - n_0\delta_{\lambda\tau}\right)^2 + K_d\sum_{i,\mu}\sigma_{i,i+\mu}$$

Fluctuations of non-contractible loops crucial - cluster/worm updates may be necessary for large system sizes



Homeier et al. 2022.

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- Generalization of mapping to non-Abelian gauge groups?